Class 12, given on Jan 29, 2010, for Math 13, Winter 2010

1. Another cylindrical coordinate example

We've seen how cylindrical coordinates were very helpful when calculating integrals over a region E shaped like a cone. In general, cylindrical coordinates are a good choice whenever the region of integration involves cylinders, cones, paraboloids, or other expressions containing $x^2 + y^2$.

Example. Consider the solid E bounded by the surfaces $z = \sqrt{x^2 + y^2}, z = 2 - x^2 - y^2$. Write down a triple integral equal to the volume of this solid, and evaluate it.

First, we begin by sketching these two surfaces. The first surface is a cone (as we have seen), while the second surface is a elliptic (even circular, in this case) paraboloid which is up-side down. Furthermore, the cone forms the bottom boundary of the surface while the paraboloid forms the top. Evidently, this solid looks somewhat like an ice-cream cone.

We want to determine the projection of E onto the xy plane. To do this, we should perhaps start by finding the intersection of the two surfaces. We set both functions of x, y equal to each other and obtain $\sqrt{x^2 + y^2} = 2 - x^2 - y^2$. If we let $r = \sqrt{x^2 + y^2}$ (which is natural as we are going to eventually end up using cylindrical coordinates anyway), this equation becomes $r = 2 - r^2$. This is a quadratic for r, and the solutions to this equation are $r = 1, -2$. We discard the $r = -2$ solution since $r \geq 0$, and so the two surfaces intersect when $r = 1$. Furthermore, when $r = 1, z = r = 2 - r^2 = 1$, so the intersection of these two surfaces is a circle given by the equations $x^2 + y^2 = 1, z = 1$.

Amongst all points in E, r is maximal at the boundary of this circle, so the projection of E onto the xy plane is the disc $x^2 + y^2 \leq 1$. In cylindrical coordinates, this corresponds to inequalities $0 \leq \theta \leq 2\pi, 0 \leq r \leq 1$. Therefore, the triple integral which equals the volume of this solid is

$$\iiint\limits_E dV = \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r \, dz \, dr \, d\theta.$$

We evaluate this integral:

$$\int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r(2-r^2-r) \, dr \, d\theta = 2\pi \left(r^2 - \frac{r^3}{3} - \frac{r^4}{4} \Big|_{r=0}^{r=1} \right) = 2\pi \left(1 - \frac{1}{3} - \frac{1}{4} \right) = \frac{5\pi}{6}.$$

Of course, we could have instead just calculated a double integral over the disc $x^2 + y^2 \leq 1$, using polar coordinates, to find this volume. However, setting up a triple integral has the advantage of letting us also calculate quantities like mass, moments, moment of inertia, etc., of a solid in the shape of E – we would not need to adjust the bounds of integration, only the integrand.

2. Spherical coordinates

We now briefly examine another coordinate system which is sometimes convenient when computing triple integrals. *Spherical coordinates* are defined in the following way: a point (x, y, z) has spherical coordinates (ρ, θ, ϕ) if

$$x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi.$$

One can check, using the identity $\sin^2\theta + \cos^2\theta = 1$, that $x^2 + y^2 + z^2 = \rho^2$; therefore, $|\rho|$ is just the distance of a point from the origin. The meaning of the other two angles is not as obvious, but do have a natural geometric interpretation.

The angle ϕ, sometimes called the *inclination angle*, measures the angle the line segment from the origin to the point in question makes with the positive z-axis. Therefore, if $\phi = 0$, the corresponding point lies somewhere on the positive z-axis; if $\phi = \pi/2 = 90^\circ$, then the point lies somewhere on the xy plane (since this plane is orthogonal to the z-axis), and if $\phi = \pi = 180^\circ$, then the point lies somewhere on the negative z-axis. With this interpretation of ϕ, it is evident that we want to restrict ϕ to lie between 0 and π.

Once we have determined ρ and ϕ, we have also determined z, but x, y are still undetermined. If you project (x, y, z) onto the xy plane, you get the point $(x, y, 0)$. Define $r = \rho \sin\phi$; then r is the distance of $(x, y, 0)$ from the origin. The angle θ, sometimes called the *azimuthal angle*, is the angle which makes $x = r\cos\theta, y = r\sin\theta$; evidently, this is the angle the line segment connecting $(x, y, 0)$ to the origin makes with the positive x-axis, just like when we considered polar or cylindrical coordinates. We want to restrict θ to lie in between 0 and 2π.

Examples.

- The graphs of equations $\rho = C, C$ some positive constant, are spheres, with center at the origin and radius ρ. This explains why these coordinates are called cylindrical coordinates.
- The graph of an equation $\phi = C$, $0 \le C \le \pi$, will in general be a cone of varying width, centered around the z-axis. If C is very small, then the cone is very thin, while if $C = \pi/2$, we actually get the xy plane instead of a cone.
- The graph of an equation $\theta = C, 0 \le \theta \le 2\pi$, is a plane which contains the z-axis, and is orthogonal to the xy plane. This is very closely related to the fact that the graphs of $\theta = C$ in polar coordinates are lines passing through the origin.
- Spherical coordinates are probably already familiar to you. The system of latitude and longitude used to identify points on the surface of the Earth are very closely related to ϕ and θ. For example, pretend the Earth is a perfect sphere and has $\rho = 1$. Then the set of points with ϕ constant form a line of longitude, since we will be looking at all points on the surface of the Earth with $z = \phi$ constant. There is a slight difference in the actual numbers used for longitude as opposed to ϕ, but this is merely cosmetic. For example, the North Pole corresponds to $\phi = 0$, which is at $90^\circ N$ longitude. The equator is 0°, which corresponds to $\phi = \pi/2$. Similarly, lines of latitude are given by θ constant. This picture of spherical coordinates as essentially being the system of latitude and longitude also allow us to see what a 'spherical rectangle' looks like. The rectangular prism $R = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2]$ is really just a rectangular prism in rectangular coordinates. In cylindrical coordinates, a 'cylindrical rectangle' takes the shape of an angular piece of the solid between two concentric cylinders. In spherical coordinates, a 'spherical rectangle' takes the shape of a solid between two concentric spheres, demarcated by some line of latitude and some line of longitude.
- What is the shape of the solid given by $0 \le \rho \le 2, 0 \le \phi \le \pi, 0 \le \theta \le 3\pi/2$? Evidently we obtain three fourths of a sphere of radius 2; the part lying over the fourth quadrant of the xy plane is cut out.
- We remark that there is no uniform notation for spherical coordinates, even to this modern day. The terminology we have chosen, with ϕ for the inclination angle and θ for the azimuth, is common amongst mathematicians and is used by the textbook. However, in other fields (physics, engineering, etc.), it is possible that the roles of

ϕ and θ are interchanged. When consulting other sources for formulas related to spherical coordinates, be sure to check which notation is being used!

What is the integration formula connecting spherical coordinates to an iterated integral? If a region E is given by spherical inequalities $0 \leq a \leq \rho \leq b, \alpha \leq \theta \leq \beta, \gamma \leq \phi \leq \delta$, then

$$\iiint\limits_E f(x,y,z)\,dV = \int_\gamma^\delta \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

In other words, we need to multiply the integrand by $\rho^2\sin\phi$ when converting to spherical coordinates. There is a similar expression for regions E which are defined by more general spherical inequalities, where some of the bounds of integrations may be functions in the variables not yet integrated, instead of constants.

Example. Use spherical coordinates to show that the volume of a sphere with radius R is equal to $4\pi R^3/3$.

A sphere of radius R, say E, is described using spherical coordinates by $0 \leq \rho \leq R, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$. Therefore, the volume of the sphere is given by the triple integral

$$\iiint\limits_E dV = \int_0^\pi \int_0^{2\pi} \int_0^R \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \left(\int_0^\pi \sin\phi\,d\phi\right)\left(\int 0^{2\pi}\,d\theta\right)\left(\int_0^R \rho^2\,d\rho\right) = 2\pi(2)\frac{R^3}{3} = \frac{4\pi R^3}{3}.$$